A risk-aversion approach for the Multiobjective Stochastic Programming problem

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Abstract

Multiobjective stochastic programming is a field well located to tackle problems arising in emergencies, given that uncertainty and multiple objectives are usually present in such problems. A new concept of solution is proposed in this work, especially designed for risk-aversion solutions. A linear programming model is presented to obtain such solution.

Keywords: Multiobjective stochastic programming; Linear programming; Risk aversion

1 Introduction

Decision making is never easy, yet we often have to make decisions. Emergencies and disaster management are fields in which many difficulties often arise, such as high uncertainty and multiple conflicting objectives. To overcome such difficulties, risk-aversion decisions are usually sought. Risk-aversion is the attitude for which we prefer to lower uncertainty rather than gambling extreme outcomes (positive or negative).

Risk-aversion, although typically studied in problems with uncertainty, can as well be considered when making decisions with multiple criteria. For instance, in the field of disaster management, solutions that are sufficiently good for all criteria are usually preferred than others that perform exceptionally good for some criteria but inadequately for the others.

Multicriteria decision making (MCDM) is a field worth of consideration when studying real-world problems. Such is the case that MCDM techniques have been recently used for solving problems as varied as: disaster management (Gutjahr and Nolz, 2016; Ferrer et al., 2018), engineering (Sun et al., 2018), finance (Karsu and Morton, 2015; Angilella and Mazzù, 2015), forest planning (Fotakis, 2015), healthcare (Guido and Conforti, 2017), location of waste facilities (Eiselt and Marianov, 2015), police districting (Liberatore and Camacho-Collados, 2016), route planning (Bast et al., 2016), train scheduling (Samà et al., 2015) or urban planning (Spina et al., 2015; Carli et al., 2018).

This situation, in which multiple conflicting objectives have to be optimized, has led to the definition of different solution concepts and methodologies. Depending on the problem and the type of solution considered, a specific methodology should be applied.

The concept of efficiency reflects the intuition that for a solution to be acceptable, another cannot exist improving that one in every objective. Multiple notions of efficiency are available. The notation that this paper follows is the given in Ehrgott (2005).

Definition 1 (Efficiency, Ehrgott (2005))**.** Let $f_1(x), \dots, f_K(x)$ be objective functions to be minimized, and let X be the feasible set. A feasible solution $\hat{x} \in X$ is called:

- *Weakly efficient* if there is no $x \in X$ such that $f(x) < f(\hat{x})$ i.e. $f_k(x) < f_k(\hat{x})$ for all $k = 1, \dots, K$.
- *Efficient* or *Pareto optimal* if there is no $x \in X$ such that $f_k(x) \leq f_k(\hat{x})$ for all $k = 1, \dots, K$ and $f_i(x) < f_i(\hat{x})$ for some $i \in \{1, \dots, K\}$.
- *Strictly efficient* if there is no $x \in X$, $x \neq \hat{x}$ such that $f(x) \leq f(\hat{x})$.

Furthermore, the set of efficient solutions is called the *efficient set*, and the image under f of this set is the *nondominated* set. It is reasonable to assume that the solution given to any problem must lie in the efficient set.

Uncertainty is another feature present in the studied problems, in which risk-averse decisions will be preferred. The most common ways for dealing with the uncertainty are *stochastic programming* and *robust optimization*, in which *fuzzy optimization* is also included (Rommelfanger, 2004).

The different approaches for treating uncertainty do not respond to the desires of the modeller, instead, they reflect the nature of the uncertainty. If the uncertainty comes with an underlying known or estimated probability distribution, then stochastic programming is used. On the other hand, if uncertainty comes from a lack of precision or semantic uncertainty, then robust optimization is used. Robust optimization does not assume a known (or existing) distribution (Ben-Tal and Nemirovski, 1999; Chen et al., 2007; Klamroth et al., 2017). A recent review of robust optimization is written in Gabrel et al. (2014). For an introduction to stochastic programming the reader is referred to Birge and Louveaux (2011).

Stochastic programming is the widest used technique when there are historical data or information to infer a probability distribution. Moreover, usually discrete distributions are used, calling scenarios the different values. The concepts of *value-at-risk* (VaR) and *conditional value-at-risk* (CVaR) are widely used for quantifying risk (see for instance Yao et al. (2013); Mansini et al. (2015); Liu et al. (2017); Dixit and Tiwari (2019); Fernández et al. (2019)). They are typically defined for losses distributions in finance, where the right tail of the distributions are of interest.

Definition 2 (CVaR, Rockafellar and Uryasev (2002))**.** Given $F_X(x)$ distribution function, and $\beta \in [0,1]$, the β-CVaR is the conditional expectancy over $\{x : F_X(x) \geq \beta\}$.

Consider now the following problem, in which multiple objectives to be minimized and uncertainty are included simultaneously:

$$\min_{x \in X} (f_1(x,\omega), \ldots, f_K(x,\omega))$$

The above problem is typically called *multiobjective stochastic programming problem* (MSP), especially if ω, the uncertainty source, has a known probability distribution.

In this paper we introduce a new solution concept in multiobjective stochastic programming based on risk-aversion preferences. Such concept is complemented with a mathematical programming model to efficiently compute it, and computational experiments are performed to assess its strengths.

Structure of paper The remaining of this paper is organized as follows. Section 2 includes the definition of a novel concept of solution for MSP problems and studies its properties. Section 2.4 illustrates a basic example of how this solution can be found if the decision space is finite and small.

Section 3 shows how to obtain such a solution with a linear programming model. An application to the multicriteria knapsack problem is developed in Section 4.

2 Multiobjective stochastic programming

2.1 Literature review

Goicoechea (1980) develops PROTRADE method, where utility functions are defined to aggregate objectives into a single objective stochastic problem. The resulting problem is solved with an interactive method, where the decision-maker defines an expected solution and a feasibility probability. Leclercq (1982) reduces the stochasticity by adding some *good* measures to the list of objectives, such as the mean, variance, or probability of being over/below a threshold. The resulting multiobjective deterministic problem is solved aggregating the objectives, but it could be solved via other techniques.

Caballero et al. (2004) compare the *stochastic approach* with the *multiobjective approach* when using different techniques. The *stochastic approach* transforms the MSP on a single-objective stochastic problem, while the *multiobjective approach* first reduces the stochasticity transforming the MSP on a deterministic multiobjective problem. They highlight that *"the multiobjective approach* forgets *the possible existence of stochastic dependencies between objectives."* Aouni et al. (2005) study stochastic goal programming, where the deviation of the objective functions to some goals set beforehand to stochastic values is minimized.

In Ben Abdelaziz and Masri (2010) a chance-constrained compromise approach is proposed, with an example presented in Ben Abdelaziz et al. (2007). In Muñoz et al. (2010) the *INTEREST* method is proposed. It is an interactive reference point method. The decision-maker gives reference levels u_i and probabilities β_i, hoping to achieve a solution x^* such that $\mathbb{P}(f_i(x^*) \leq u_i) \geq \beta_i$. If this is infeasible, the decision-maker should either increase the reference levels or decrease the probabilities of achievement. Ben

Abdelaziz (2012) reviews different solutions methods for the MSP problem, categorizing them as stochastic approach or multiobjective approach.

Some fields where MSP models have been developed are: forest management (Álvarez-Miranda et al., 2018), multiple response optimization (Díaz-García and Bashiri, 2014), energy generation (Teghem et al., 1986; Bath et al., 2004), energy exchange (Gazijahani et al., 2018), capacity investment (Claro and De Sousa, 2010), disaster management (Manopiniwes and Irohara, 2016; Bastian et al., 2016), portfolio optimization (Şakar and Köksalan, 2012) and cash management (Salas-Molina et al., 2019), among others

2.2 Definitions and dominance relationship

The concept of CVaR allows to aggregate several scenarios by just looking at what happens in the worst ones. The *ordered weighted averaging* (OWA) operators are defined in Yager (1988), and independently in the field of locational analysis Carrizosa et al. (1994); Nickel and Puerto (1999) under the name of *ordered median function.* These concepts will allow us to aggregate different criteria by looking at the least desirable ones, as a risk-aversion measure.

Definition 3 (OWA, Yager (1988))**.** Given $a_1, \dots, a_n \in \mathbb{R}$, the *ordered weighted averaging* (OWA) operator with weights $\lambda_1, \dots, \lambda_n$ is defined as:

$$OWA(a_1, \dots, a_n) = \sum_i \lambda_i a_{(i)}$$

where $\left(a_{(1)}, \dots, a_{(n)}\right)$ is the ordered vector from largest to smallest $(a_1, \dots, a_n)$.

Remark 1*.* For certain weights, the OWA represents a known quantity:

- If $\lambda_i = \frac{1}{n}$, the resulting OWA is the average of a.
- If $\lambda_1 = 1$, and $\lambda_j = 0$ for $j > 1$, the OWA is the maximum of a.
- If $\lambda_n = 1$, and $\lambda_j = 0$ for $j < n$, the OWA is the minimum of a.

Yager and Alajlan (2016) later study how to assign weights for an OWA when criteria have different importances.

Definition 4 (OWA with importances, Yager and Alajlan (2016))**.** Given $a_1, \dots, a_n \in \mathbb{R}$ with importances $u_1, \dots, u_n$ such that $\sum_i u_i = 1$ the weights λ_j for the OWA can be calculated with f, the *weight generating function* in the following manner:

1. Sort vector a such that $a_{(1)} \geq a_{(2)} \geq \dots \geq a_{(n)}$.
2. With $(\cdot)$ as the order induced by a, define $T_j = \sum_{k=1}^{j} u_{(k)}$.
3. Let f be a function such that $f : [0, 1] \to [0, 1]$ and $f(0) = 0, f(1) = 1$. This function is called *weight generating function.*
4. Obtain the weights as $\lambda_j = f(T_j) - f(T_{j-1})$.

Example 1 (of Definition 4)**.** Consider the following weight generating function, for a given $r \in (0, 1]$:

$$f(x) = \begin{cases} \frac{x}{r} & \text{if } x < r \\ 1 & \text{if } x \geq r \end{cases}$$

Let $(\cdot)$ be the order such that $a_{(1)} \geq \cdots \geq a_{(n)}$, $u_{(j)}$ the weight associated to $a_{(j)}$, and also let $T_j = \sum_{k=1}^{j} u_{(k)}$. We shall see know how the weights are obtained from f. Let j^* be such that $T_{j^*-1} < r \leq T_{j^*}$.

- $\lambda_1 = f(T_1) = f(u_{(1)}) = \frac{u_{(1)}}{r}$, assuming $u_{(1)} < r$
- $\lambda_2 = f(T_2) - f(T_1) = f(u_{(1)} + u_{(2)}) - f(u_{(1)}) = \frac{u_{(1)}+u_{(2)}}{r} - \frac{u_{(1)}}{r} = \frac{u_{(2)}}{r}$, assuming $u_{(1)} + u_{(2)} < r$
- $\ldots$
- $\lambda_{j^*} = f(T_{j^*}) - f(T_{j^*-1}) = 1 - \left(\frac{u_{(1)}+u_{(2)}+\cdots+u_{(j^*-1)}}{r}\right)$, since $T_{j^*} \geq r$
- $\lambda_{j^*+1} = f(T_{j^*+1}) - f(T_{j^*}) = 1 - 1 = 0$
- $\ldots$
- $\lambda_n = f(T_n) - f(T_{n-1}) = 1 - 1 = 0$

Consequently the OWA of $a_1, \ldots, a_n$ with importances $u_1, \ldots, u_n$ is:

$$\begin{aligned} OWA =& \frac{u_{(1)}}{r} a_{(1)} + \frac{u_{(2)}}{r} a_{(2)} + \cdots + \left[1 - \left(\frac{u_{(1)} + u_{(2)} + \cdots + u_{(j^*-1)}}{r}\right)\right] a_{(j^*)} \\ =& \frac{u_{(1)} a_{(1)} + u_{(2)} a_{(2)} + \cdots + \left(r - u_{(1)} - u_{(2)} - \ldots\right) a_{(j^*)}}{r} \end{aligned}$$

That is, the OWA is the average of the worst a_j, weighted by their importances, with total importance adding up to r

The starting point of this paper is the recurrent idea of representing ordered weighted or ordered median operators by means of k-sums. k-sums (or k-centra in the location analysis literature) are sums of the k-largest terms of a vector (Puerto et al., 2017). One can trace back, at least to Kalcsics et al. (2002), the use of k-sums to represent ordered median objectives. More recent references are for instance Blanco et al. (2013, 2014); Ponce et al. (2018) and Filippi et al. (2019). This last reference introduces a normalized version of k-centrum, named β-average that will be used in our paper.

Through the remaining of the paper consider that $f_k^j(x)$ are functions to be minimized within a feasible set X, with $k = 1, \ldots, K$ representing K different objectives with importances w_k and $j = 1, \ldots, J$ encoding J different scenarios with probabilities π_j.

Definition 5 (β-average, $g_k^\beta(x)$, Filippi et al. (2019))**.** Given $\beta \in (0, 1]$, for each criterion k it can be defined $g_k^\beta(x)$ which measures the average of f on the worst scenarios $\left(f_k^1(x), \ldots, f_k^J(x)\right)$, with accumulated probability equal to β.

Remark 2 (Filippi et al. (2019)). Given a value β, if the sum of the probabilities of the worst scenarios is exactly β, then the β-average is exactly $(1-\beta)$-CVaR.

Example 2. Consider a point x, a fixed criterion k and 5 different scenarios with probabilities π_j and values of f_k^j given. Table 1 shows the β-averages for different values of β, in which the scenarios have been ordered from largest value of f to smallest.

- For $\beta = 0.2$, the scenario $j = 1$ is the only one needed to obtain the worst scenario with probability 0.2, and hence $g_k^\beta(x) = \frac{0.2 \cdot 10}{0.2} = 0.2$.
- When β equals 0.3 it is necessary to include scenario 2, obtaining a β-average of $\frac{0.2 \cdot 10 + 0.1 \cdot 7}{0.3} = 9$.
- Finally if $\beta = 0.5$ scenario 3 needs to be added as well, but only with the probability needed until reaching 0.5: $g_k^\beta(x) = \frac{0.2 \cdot 10 + 0.1 \cdot 7 + 0.2 \cdot 4}{0.5} = 7$.

Table 1: Small example of β-average for different values of β

	scenario					β		
	1	2	3	4	5	0.2	0.3	0.5
π_j	0.2	0.1	0.3	0.25	0.15	10	9	7
$f_k^j(x)$	10	7	4	3	2			

When using the β-average the functions $f_k^j(x)$ were transformed into $g_k^\beta(x)$, a collection of K functions not depending on the scenario. An OWA will be defined now, via its weight generating function, that will reduce the K β-averages into a scalar function.

Definition 6 (r-OWA, $O_r(x)$)**.** Given $x_i \in \mathbb{R}$ with importance w_i ($i = 1, \ldots, K$, $w_i \geq 0, \sum_i w_i = 1$) and $r \in (0, 1]$, the function $O_r(x)$ is defined as the OWA with the following weight generating function:

$$f(x) = \begin{cases} \frac{x}{r} & \text{if } x < r \\ 1 & \text{if } x \geq r \end{cases}$$

Remark 3. The definition of $O_r(x)$ is made on a similar manner that the one given of the β-average (Definition 5), but it is done on a context with importances rather than probabilities. Example 3 shows the similarities between both approaches.

Example 3. Consider a point x and let $g_k(x)$ be the evaluation of x under 5 different criteria with importances w_j. Table 2 shows the r-OWAs for different values of r, in which the criteria have been ordered from largest values of $g_k(x)$ to smallest. Consider the case $r = 0.5$:

1. As $g_k(x)$ are already ordered for largest to smallest, the values of T_k are:

$$T_1 = 0.2, T_2 = 0.2 + 0.1 = 0.3, T_3 = 0.6, T_4 = 0.85, T_5 = 1$$

2. The values of T_k under f:

$$f(T_1) = \frac{0.2}{0.5}, f(T_2) = \frac{0.3}{0.5}, f(T_3) = f(T_4) = f(T_5) = 1$$

3. The weights of the OWA:

$$\lambda_1 = \frac{0.2}{0.5}, \lambda_2 = \frac{0.3 - 0.2}{0.5} = \frac{0.1}{0.5}, \lambda_3 = 1 - \frac{0.3}{0.5} = \frac{0.2}{0.5}, \lambda_4 = \lambda_5 = 0$$

4. Consequently the r-OWA is:

$$r\text{-OWA} = \frac{0.2x_{(1)} + 0.1x_{(2)} + 0.2x_{(3)}}{0.5} = \frac{0.2 \cdot 10 + 0.1 \cdot 7 + 0.2 \cdot 4}{0.5} = 7$$

Table 2: Small example of r-OWA for different values of r

	criterion					r		
	1	2	3	4	5	0.2	0.3	0.5
w_k	0.2	0.1	0.3	0.25	0.15	10	9	7
$g_k(x)$	10	7	4	3	2			

Remark 4. Given $x_1, \ldots, x_K$ and its associated importances $w_1, \ldots, w_K$, then the λ_k of the r-OWA are determined in such a way that:

$$O_r(x) = \max\left\{ \frac{\tilde{\lambda}_1 x_1 + \cdots + \tilde{\lambda}_K x_K}{r} \mid \tilde{\lambda}_k \leq w_k, \sum \tilde{\lambda}_k = r \right\} \quad \text{with } \lambda_k = \frac{\tilde{\lambda}_k}{r}$$

Given $r, \beta \in (0, 1]$ and $x \in X$, let us introduce the function $h_r^\beta(x)$ as the r-OWA of the β-averages. That is:

$$h_r^\beta(x) = O_r\left(g_1^\beta(x), \ldots, g_K^\beta(x)\right)$$

Remark 5. If the importance of all criteria is the same ($w_k = \frac{1}{K}$ for all k) and $r = \frac{n}{K}$ with $n \in \{1, \ldots, K\}$, then the $h_r^\beta(x)$ is the average of the n worst β-averages. Recall that this is called n-centra (Nickel and Puerto, 2005).

Definition 7 (Dominance)**.** Let x and y feasible solutions ($x, y \in X$) and $r, \beta \in (0, 1]$. Then x dominates y ($x \succsim y$) if $h_r^\beta(x) \leq h_r^\beta(y)$, where $h_r^\beta(x)$ is the r-OWA of the β-averages.

Definition 7 induces a domination relationship with the following properties:

Reflexivity Given x, $h_r^\beta(x) \geq h_r^\beta(x)$, and then $x \succsim x$, so $\succsim$ is reflexive.

Transitiveness Given $x \succsim y$, $y \succsim z$, we have $h_r^\beta(x) \geq h_r^\beta(y)$ y $h_r^\beta(y) \geq h_r^\beta(z)$, and then $h_r^\beta(x) \geq h_r^\beta(z)$, which leads to $x \succsim z$, and we conclude that $\succsim$ is transitive.

Antisymmetry Given $x \succsim y$, $y \succsim x$, we have $h_r^\beta(x) \geq h_r^\beta(y)$ and $h_r^\beta(y) \geq h_r^\beta(x)$, but from $h_r^\beta(x) = h_r^\beta(y)$ it cannot be guaranteed that $x = y$, and hence $\succsim$ is not antisymmetric.

2.3 Idea of solution and dominance properties

Consider the multiobjective stochastic programming problem:

$$\min_{x \in X} (f_1(x,\omega), \ldots, f_K(x,\omega))$$

The previously defined concepts of β-average and r-OWA transform the MSP problem into a deterministic multiple objective problem, and then into a deterministic single objective problem.

$$MSP \to MOP \to LP(MIP)$$

$$f_k^j(x) \xrightarrow{\beta\text{-average}} g_k^\beta(x) \xrightarrow{r\text{-OWA}} h_r^\beta(x)$$

1. For every $x \in X$ there is a function f_k^j to be minimized which depends on the scenario j and the criterion k.
2. The problem is transformed into a deterministic one with multiple objectives (MOP) using the β-average concept.
3. Computing the r-OWA, each $x \in X$ is assigned a scalar. The problem consists of finding the x which minimizes this $h_r^\beta(x)$.

The solution procedure lies into what is usually called a *scalarization approach.* When obtaining a minimizer of $h_r^\beta(x)$ it is also desired that the optimal solution is efficient for the associated MOP problem:

$$\min_{x \in X} \left(g_1^\beta(x), \ldots, g_K^\beta(x)\right) \tag{MOP}$$

Proposition 1. *Given x^* minimum of $h_r^\beta(x)$ the following statements hold:*

1. *x^* is not necessarily efficient of the MOP problem.*
2. *x^* is weakly efficient of the MOP problem.*
3. *If x^* is the only minimum of $h_r^\beta(x)$, then x^* is efficient.*
4. *Given x^* not efficient, an alternative y^* can be found on a second phase such that y^* is efficient and $h_r^\beta(x^*) = h_r^\beta(y^*)$.*

These properties are known when using scalarization techniques (Ehrgott, 2005). Hence only an example of the first statement will be shown.

Example 4 (x^* is not necessarily efficient)**.** Consider the example displayed on Table 3, in which there are only two feasible solutions, two equiprobable scenarios ($\pi_1 = \pi_2 = \frac{1}{2}$), three equally important criteria ($w_1 = w_2 = w_3 = \frac{1}{3}$), and consider the values of $\beta = \frac{1}{2}$ and $r = \frac{2}{3}$ are taken.

The β-averages are $(0.8, 0.4, 0.65)$ for the first alternative and $(0.8, 0.45, 0.65)$ for the second alternative. When computing the function h_r^β, both alternatives have an objective

Table 3: Values of two alternatives for each scenario j and criterion k, together with their β-averages ($\beta = \frac{1}{2}$) and r-OWAs ($r = \frac{2}{3}$)

Table 4: Alternative 1

	k_1	k_2	k_3
j_1	0.80	0.40	0.30
j_2	0.60	0.20	0.65
β-average	0.80	0.40	0.65
r-OWA		0.725	

Table 5: Alternative 2

	k_1	k_2	k_3
j_1	0.70	0.45	0.65
j_2	0.80	0.30	0.50
β-average	0.80	0.45	0.65
r-OWA		0.725	

value of 0.725. Consequently, even though the second alternative is an optimal solution of h_r^β, it is not an efficient solution of the MOP problem as its β-averages are dominated by those of the first alternative.

2.4 An illustrative example

The solution concept proposed will be now applied, first with a discrete (and small) case. When the solution space is discrete, and all feasible solutions can be explicitly enumerated, the steps are as follows:

Step 0 Normalize all objective functions $f_k^j(x)$.

Step 1 Set values for $\beta, r \in (0, 1]$.

Step 2 For every $x \in X$ and every criterion define $g_k^\beta(x)$ as:

$$g_k^\beta(x) = \begin{array}{l} \textit{average of worst scenarios for criterion } k \\ \textit{with probabilities adding up to } \beta \end{array}$$

Step 3 Define $h_r^\beta(x)$ as:

$$h_r^\beta(x) = \begin{array}{l} \textit{average of worst } g_k^\beta(x) \textit{ values} \\ \textit{with importances adding up to } r \end{array}$$

Step 4 Search for $x \in X$ minimizing $h_r^\beta(x)$

Assume a decision space with only four alternatives, evaluated under five different scenarios with six criteria. For each of those alternatives it can be computed the value of the functions $f_k^j(x)$ to be minimized. Table 6 shows the values of f, evaluated on feasible point x_1, by each of the scenarios and criteria considered.

The first step consists on calculating the β-averages. Let assume a value of $\beta = 0.3$:

1. For the first criterion the worst scenario is j_5, which has probability 0.1. The second worst is j_4, with a probability of 0.25. As the sum of those probabilities exceeds the β fixed, for computing the β-average just a probability of 0.2 is considered:

$$g_1^\beta(x_1) = \frac{0.1 \cdot 0.86 + 0.2 \cdot 0.76}{0.3} = 0.793$$

Table 6: Values of alternative 1 by scenario (j) and criteria (k)

			criteria					
			$w_1 = 0.20$	$w_2 = 0.10$	$w_3 = 0.20$	$w_4 = 0.25$	$w_5 = 0.15$	$w_6 = 0.10$
			k_1	k_2	k_3	k_4	k_5	k_6
scenarios	$\pi_1 = 0.15$	j_1	0.51	0.27	0.39	0.45	0.75	0.76
	$\pi_2 = 0.20$	j_2	0.58	0.65	0.47	0.26	0.90	0.24
	$\pi_3 = 0.30$	j_3	0.48	0.44	0.90	0.50	0.93	0.65
	$\pi_4 = 0.25$	j_4	0.76	0.18	0.01	0.90	0.56	0.02
	$\pi_5 = 0.10$	j_5	0.86	0.36	0.21	0.28	0.63	0.72

2. $g_2^\beta(x_1) = (0.2 \cdot 0.65 + 0.1 \cdot 0.44)/0.3 = 0.580$
3. $g_3^\beta(x_1) = (0.3 \cdot 0.90)/0.3 = 0.900$
4. $g_4^\beta(x_1) = 0.833$, $g_5^\beta(x_1) = 0.930$, $g_6^\beta(x_1) = 0.728$

The last step is calculating the function $h_r^\beta(x)$, that is, the r-OWA of the β-averages. Table 7 calculates the r-OWA, and shows as well the information of the previously calculated β-averages, when the value of $r = 0.17$ is taken.

Table 7: Values of alternative 1 by scenario (j) and criteria (k)

			criteria					
			$w_1 = 0.20$	$w_2 = 0.10$	$w_3 = 0.20$	$w_4 = 0.25$	$w_5 = 0.15$	$w_6 = 0.10$
			k_1	k_2	k_3	k_4	k_5	k_6
scenarios	$\pi_1 = 0.15$	j_1	0.51	0.27	0.39	0.45	0.75	0.76
	$\pi_2 = 0.20$	j_2	0.58	0.65	0.47	0.26	0.90	0.24
	$\pi_3 = 0.30$	j_3	0.48	0.44	0.90	0.50	0.93	0.65
	$\pi_4 = 0.25$	j_4	0.76	0.18	0.01	0.90	0.56	0.02
	$\pi_5 = 0.10$	j_5	0.86	0.36	0.21	0.28	0.63	0.72
β-average, $\beta = 0.30$			0.793	0.580	0.900	0.833	0.930	0.728
r-OWA, $r = 0.17$			0.927					

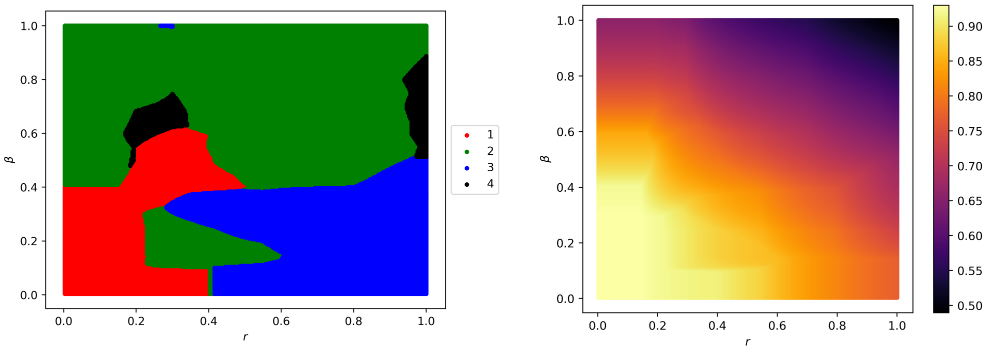
Results The values of the functions for the other alternatives, as well as its β-averages and r-OWAs are shown in Tables 13, 14 and 15, starting on Page 27. A summary of the results can be seen in Table 8, where all the β-averages and r-OWAs are shown, determining that the optimal alternative for the values of β and r given is Alternative 1.

Table 8: β-averages and r-OWAs for each of the 4 feasible alternatives of the example

	β-averages						r-OWA
	$g_1^\beta(x)$	$g_2^\beta(x)$	$g_3^\beta(x)$	$g_4^\beta(x)$	$g_5^\beta(x)$	$g_6^\beta(x)$	$h_r^\beta(x)$
Alternative 1	0.793	0.580	0.900	0.833	0.930	0.728	0.927
Alternative 2	0.930	0.832	0.703	0.820	0.660	0.770	0.930
Alternative 3	0.765	0.775	0.468	0.643	0.950	0.883	0.943
Alternative 4	0.993	0.760	0.473	0.773	0.820	0.990	0.993

Variations on β and r yield very different results. Figure 1a shows which of the four alternatives has the lowest h value, depending on the values of β and r.

Figure 1b shows the optimal objective value when varying the parameters β and r. It can be appreciated how h decreases when β and r increase. This is due to the fact that the original f_k^j functions are to be minimized, and the larger the parameters β and r are, more favourable scenarios/criteria will take part on the computation of $h_r^\beta(x)$, hence decreasing its optimal value.



(a) Optimal alternative for some values of r and β

(b) Optimal values of function $h_r^\beta(x)$ for some values of r and β

Figure 1: Results from illustrative example

3 Computing the minimum: continuous case

A concept of solution was proposed with Definition 7. When the functions $f_k^j(x)$ to be minimized are given, a new function $h_r^\beta(x)$ to be minimized is defined, with parameters β and r such that $h_r^\beta(x)$ is the r-OWA of the β-averages. If the decision space is sufficiently small, the procedure shown in the above example obtains such a solution.

In this section, a mathematical programming model will be developed to obtain the minimum of $h_r^\beta(x)$ which allows one to obtain the proposed solution for bigger decision spaces, including continuous ones.

3.1 Mathematical programming model

Given k and $x \in X$ we have the vector $\left(f_k^1(x), \ldots, f_k^J(x)\right)$. Let $\left(f_k^{(1)}(x), \ldots, f_k^{(J)}(x)\right)$ be the ordered vector such that $f_k^{(j_1)}(x) \geq f_k^{(j_2)}(x)$ when $j_1 \leq j_2$.

Given $\beta \in (0, 1]$, let $\hat{\jmath}$ be the ordered scenario such that:

$$\sum_{j=1}^{\hat{\jmath}} \pi_{(j)} \geq \beta, \qquad \sum_{j=1}^{\hat{\jmath}-1} \pi_{(j)} < \beta$$

Alternatively:

$$
\begin{gathered}
f_k^{(1)}(x) \geq f_k^{(2)}(x) \geq \cdots \geq f_k^{(\hat{\jmath})}(x) \geq f_k^{(\hat{\jmath}+1)}(x) \geq \cdots \geq f_k^{(J)}(x) \\
1 = \underbrace{\underbrace{\pi_{(1)} + \pi_{(2)} + \cdots + \pi_{(\hat{\jmath}-1)}}_{<\beta} + \pi_{(\hat{\jmath})}}_{\geq \beta} + \cdots + \pi_{(J)}
\end{gathered}
$$

Also let:

$$
\hat{\pi}_j = \begin{cases} \pi_j & j \in \{(1), \ldots, (\hat{\jmath}-1)\} \\ \beta - \sum_{j=(1)}^{j=(\hat{\jmath}-1)} \pi_j & j = \hat{\jmath} \\ 0 & \text{otherwise} \end{cases}
$$

The definition of $\hat{\pi}_{\hat{\jmath}}$ is made in such a way that $\sum_j \hat{\pi}_j = \beta$. In this way, the average of the β worst values can be computed as $\frac{1}{\beta}\sum_{j=1}^{J} \hat{\pi}_j f_k^{(j)}(x)$, which coincides with the definition of β-average (Definition 4). This computation can be written as the following optimization problem:

$$
\begin{aligned}
\max_{\tilde{u}_j} \quad & \frac{1}{\beta}\sum_{j=1}^{J} \tilde{u}_j \cdot f_k^j(x) \\
\text{s.t.} \quad & \sum_{j=1}^{J} \tilde{u}_j = \beta \\
& 0 \leq \tilde{u}_j \leq \pi_j \qquad j = 1, \ldots, J
\end{aligned}
$$

A more natural approach would be to consider $u_j = \frac{\tilde{u}_j}{\beta}$. These u_j represent the proportion in which scenario j plays a part on the aggregated β-average. Introducing that change, the model is:

$$
\begin{aligned}
\max_{u_j} \quad & \sum_{j=1}^{J} u_j \cdot f_k^j(x) \\
\text{s.t.} \quad & \sum_{j=1}^{J} u_j = 1 \\
& 0 \leq u_j \leq \frac{\pi_j}{\beta} \qquad j = 1, \ldots, J
\end{aligned}
$$

The dual formulation is:

$$
\begin{aligned}
\min_{z,y_j} \quad & z + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_j \\
\text{s.t.} \quad & z + y_j \geq f_k^j(x) \qquad j = 1, \ldots, J \\
& z \text{ free}, y_j \geq 0
\end{aligned} \tag{1}
$$

And hence finding the $x \in X$ which minimizes the average of the worst β scenarios for a given k is:

$$\min_{x\in X} \left(\begin{array}{lll} \max_{\tilde{u}_j} & \frac{1}{\beta}\sum_{j=1}^{J} \tilde{u}_j f_k^j(x) & \\ \text{s.t.} & \sum_{j=1}^{J} \tilde{u}_j = \beta & \\ & 0 \leq \tilde{u}_j \leq \pi_j & j = 1,\ldots,J \end{array} \right)$$

Or alternatively:

$$\min_{x\in X} \left(\begin{array}{lll} \min_{z,y_j} & z + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_j & \\ \text{s.t.} & z + y_j \geq f_k^j(x) & j = 1,\ldots,J \\ & z \text{ free}, y_j \geq 0 & j = 1,\ldots,J \end{array} \right) \tag{2}$$

Which is equivalent to:

$$\min_{z,y_j,x} \quad z + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_j \tag{3a}$$

$$\text{s.t.} \quad z + y_j \geq f_k^j(x) \quad j = 1,\ldots,J \tag{3b}$$

$$z \text{ free}, y_j \geq 0 \quad j = 1,\ldots,J$$

$$x \in X$$

Remark 6. Models (2) and (3) are equivalent, as for any $x \in X$ chosen in (3) the values z and y_j will get as small as allowed by constraint (3b), as this improves the objective function (3a). Consequently for every x, its β-average will be computed appropriately, and thus (3) obtains the $x \in X$ with smallest β-average, as desired on (2).

For every $k \in \{1,\ldots,K\}$ thanks to the problem (1) the function $g_k^\beta(x)$ can be defined, which measures for each $x \in X$ the β-average for that criterion, being:

$$\begin{aligned} g_k^\beta(x) \equiv \min_{z_k,y_{kj}} z_k + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_{kj} & \\ \text{s.t. } z_k + y_{kj} \geq f_k^j(x) & \quad j = 1,\ldots,J \\ z_k \text{ free}, y_{kj} \geq 0 & \quad j = 1,\ldots,J \end{aligned} \tag{4}$$

The already known approach for single criterion problems ends here. Given that, the next step is finding a "good" solution for all k. That is:

$$\min_{x\in X} \left(g_1^\beta(x),\ldots,g_K^\beta(x) \right)$$

Given $r \in (0,1]$ the r-OWA of the β-averages will be now computed (in accordance with the definition given in Section 2). That is, the solution of the following problem is sought:

$$\begin{aligned} \max_{\tilde{t}_k} \quad & \frac{1}{r}\sum_k \tilde{t}_k \cdot g_k^\beta(x) \\ & \sum_k \tilde{t}_k = r \\ & 0 \leq \tilde{t}_k \leq w_k \qquad k = 1,\ldots,K \end{aligned}$$

Or equivalently:

$$\begin{aligned} \max_{t_k} \quad & \sum_k t_k \cdot g_k^\beta(x) \\ & \sum_k t_k = 1 \\ & 0 \le t_k \le \frac{w_k}{r} \qquad k = 1, \dots, K \end{aligned}$$

Its dual formulation is:

$$\begin{aligned} \min_{z, v_k} \quad & z + \sum_k \frac{w_k}{r} v_k \\ \text{s.t.} \quad & z + v_k \ge g_k^\beta(x) \quad k = 1, \dots, K \\ & z \text{ free}, v_k \ge 0 \qquad k = 1, \dots, K \end{aligned}$$

Replacing the value of $g_k^\beta(x)$ given in (4) the next model is obtained:

$$\min_{z, v_k} \quad z + \sum_k \frac{w_k}{r} v_k \tag{5a}$$

$$\text{s.t.} \quad z + v_k \ge \begin{pmatrix} \min\limits_{z_k, y_{kj}} & z_k + \sum_{j=1}^J \frac{\pi_j}{\beta} y_{kj} & \\ \text{s.t.} & z_k + y_{kj} \ge f_k^j(x) & \forall j \\ & z_k \text{ free}, y_{kj} \ge 0 & \end{pmatrix} \forall k \tag{5b}$$

$$z \text{ free}, v_k \ge 0 \qquad \forall k \tag{5c}$$

Model (5) calculates for a given $x \in X$ the r-OWA of its β-averages, which coincides with the notion of the function $h(x)$ given in Section 2. This problem is not explicit in that it contains nested optimization problems in the constraints. For that reason, we propose a single level alternative for $x \in X$ fixed.

Consider the following linear programming model:

$$\min_{z, v_k, z_k, y_{kj}} \quad z + \sum_k \frac{w_k}{r} v_k \tag{6a}$$

$$\text{s.t.} \quad z + v_k \ge z_k + \sum_{j=1}^J \frac{\pi_j}{\beta} y_{kj} \qquad \forall k \tag{6b}$$

$$z_k + y_{kj} \ge f_k^j(x) \qquad \forall k, j \tag{6c}$$

$$y_{kj} \ge 0 \qquad \forall k, j \tag{6d}$$

$$z_k \text{ free}, v_k \ge 0 \qquad \forall k \tag{6e}$$

$$z \text{ free} \tag{6f}$$

Proposition 2. *Transformation from model (5) to model (6) is valid, in that their optimal solution and objective values coincide.*

Proof. Let $\left(z^*, v_k^*, z_k^*, y_{kj}^*\right)$ be the optimal solution of model (6). (z^*, v_k^*) is feasible of model (5), and it will be shown that it is also optimal for such model. Assume it exists (z', v_k') feasible of model (5) with:

$$z' + \sum_k \frac{w_k}{r} v_k' < z^* + \sum_k \frac{w_k}{r} v_k^*$$

This and constraint (6b) implies there exists k_0 such that:

$$z' + v_{k_0}' < z_{k_0}^* + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_{k_0 j}^*$$

otherwise $\left(z', v_k', z_k^*, y_{kj}^*\right)$ would be optimal of model (6). Since $z_{k_0}^*$ and $y_{k_0 j}^*$ are feasible of model (6) they are also feasible of the model on the RHS of constraint (5b), and thus z' and v_{k_0}' violate constraint (5b). □

Proposition 2 showed that the optimal solutions of models (5) and (6) coincide. Proposition 3 goes further showing the connection between their feasible sets.

Proposition 3. *The feasible set of model (5) is a projection of the feasible set of model (6).*

Proof.

1. For each feasible solution (z, v_k) of model (5) there is at least one feasible solution of model (6) with same values (z, v_k), being so the same objective function.

   Let (z^1, v_k^1) a feasible solution of model (5), and (z_k^*, y_{kj}^*) the optimal solution where the minimum of the right-hand side of equation (5b) is reached for each k. Since constraints (6b), (6c), (6d) and (6e) are satisfied in model (5), $(z^1, v_k^1, z_k^*, y_{kj}^*)$ is a feasible solution or model (6).

2. For each feasible solution (z, v_k, z_k, y_{kj}) of model (6), (z, v_k) is a feasible solution of model (5), being so the same objective function. Let $(z^2, v_k^2, z_k^2, y_{kj}^2)$ a feasible solution of model (6). Since constraints (6b), (6c) and (6d) are included in model (6), (z_k^2, y_{kj}^2) is feasible for the model included in the RHS of constraint (5b) and therefore greater than or equal to the minimum of that model, verifying:

$$z^2 + v_k^2 \geq z_k^2 + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_{kj}^2 \geq \min \left\{ z_k + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_{kj} \right\}$$

   and so, feasible for model (5).

□

Finally after proving the validity of model (6) it is possible to let $x \in X$ free, with the purpose of finding the one minimizing the function $h_r^\beta(x)$:

$$\begin{aligned}
\min_{z,v_k,z_k,y_{kj},x} \quad & z + \sum_k \frac{w_k}{r} v_k \\
\text{s.t.} \quad & z + v_k \geq z_k + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_{kj} && \forall k \\
& z_k + y_{kj} \geq f_k^j(x) && \forall k, j \\
& y_{kj} \geq 0 && \forall k, j \\
& z_k \text{ free}, v_k \geq 0 && \forall k \\
& z \text{ free} \\
& x \in X
\end{aligned}$$

4 Knapsack problem

The multiobjective stochastic knapsack problem is used to illustrate the usefulness of the previously defined concept.

Definition 8 (Multiobjective stochastic knapsack problem)**.** Let I be a collection of objects with weights v_i, which can be selected as members of a knapsack with maximum weight V. There is a set of scenarios J, each of them with probability π_j, and a set of criteria K, with importances w_k. For every pair of scenario-criterion, there is a benefit associated with selecting object i, denoted by b_{jk}^i. Which objects should be taken in order to maximize benefit?

The above problem differs with the well-known knapsack problem in that there is stochasticity and multiple objectives to be maximized.

The following MSP model can be adapted to analyze the problem. Note that to preserve the sense of the optimization, rather than to maximize the benefits of the carried objects, it will be minimized the value of the objects not chosen.

$$\begin{aligned}
\min_{x_i} \quad & \left\{ f_k^j(\mathbf{x}) := \sum_i (1 - x_i)\, b_{kj}^i \right\} \\
\text{s.t.} \quad & \sum_i v_i x_i \leq V && \forall i \\
& x_i \in \{0, 1\} && \forall i
\end{aligned} \tag{7}$$

When using the methodology developed in the previous sections, problem (7) is transformed into the following mixed-integer linear programming model:

$$
\begin{aligned}
\min_{z,v_k,z_k,y_{kj},x_i} \quad & z + \sum_k \frac{w_k}{r} v_k & \\
\text{s.t.} \quad & z + v_k \geq z_k + \sum_{j=1}^{J} \frac{\pi_j}{\beta} y_{kj} & \forall k \\
& z_k + y_{kj} \geq \sum_i (1 - x_i)\, b^i_{kj} & \forall k, j \\
& \sum_i v_i x_i \leq V & \forall i \\
& y_{kj} \geq 0 & \forall k, j \\
& x_i \in \{0, 1\} & \forall i \\
& z_k \text{ free}, v_k \geq 0 & \forall k \\
& z \text{ free} &
\end{aligned}
\tag{MSP}
$$

For given $r, \beta \in (0, 1]$, model (MSP) obtains the $\mathbf{x}^*$ minimizing the r-OWA of the β-averages. In order to illustrate the benefits of using model (MSP), a naive way of solving problem (7) is considered:

$$
\begin{aligned}
\min_{x_i} \quad & \sum_{k,j} w_k \pi_j \sum_i (1 - x_i)\, b^i_{kj} & \\
\text{s.t.} \quad & \sum_i v_i x_i \leq V & \forall i \\
& x_i \in \{0, 1\} & \forall i
\end{aligned}
\tag{MIP}
$$

Hence model (MIP) computes the average of the f^j_k, using the importances of the criteria and the probability of the scenarios. It is clear that for "average" criteria-scenarios x^*_{MIP}, the optimal solution of model (MIP), outperforms x^*_{MSP}, the optimal solution of model (MSP). Conversely x^*_{MSP} will improve x^*_{MIP} in unfavourable criteria-scenarios, as expected of a risk-averse solution.

4.1 Computational experiments

The following sections will show computational experiments, for different values of r and β and different number of objects, scenarios and criteria. Algorithm 1 shows how the random instances are created, given a number of objects, scenarios and criteria.

For each of the solved instances it will be recorded:

- $t_{\text{MSP}}, t_{\text{MIP}}$: Solution time in seconds of models (MSP) and (MIP). With them the following value is calculated:

  $$\Delta_{\text{time}} := \frac{t_{\text{MSP}}}{t_{\text{MIP}}} \qquad \textit{(time penalty factor)}$$

  Δ_{time}, the time penalty factor, indicates the increase of computing time when solving model (MSP) rather than model (MIP).

- $z^*_{\text{MSP}}, z^*_{\text{MIP}}$: Optimal values of the models.

Algorithm 1 Generating random data, with $\mathcal{U}(a,b)$ the uniform distribution in $[a,b]$

```
function RANDOMINSTANCE(|I|, |J|, |K|)
    p ← U(0.25, 0.75)            ▷ how many objects on average will fit in the knapsack
    W ← 1/p                      ▷ average weight of objects
    for i ∈ I do
        w_i ← U(0.5W, 1.5W)      ▷ weight of each object
        for j, k ∈ J × K do
            b^i_kj ← U(0, 1)     ▷ value of each object
        end for
    end for
end function
```

- $f_{\text{MSP}}\left(x^*_{\text{MIP}}\right), f_{\text{MIP}}\left(x^*_{\text{MSP}}\right)$: Objective value of x^*_{MIP} in model (MSP) and vice versa.
- To grasp the difference between the MSP and the naive approach, the following will be calculated:

$$\Delta_{\text{avg}} := 100\frac{f_{\text{MIP}}\left(x^*_{\text{MSP}}\right) - z^*_{\text{MIP}}}{z^*_{\text{MIP}}} \qquad \textit{(deteriorating rate)}$$

$$\Delta_{\text{tail}} := 100\frac{f_{\text{MSP}}\left(x^*_{\text{MIP}}\right) - z^*_{\text{MSP}}}{f_{\text{MSP}}\left(x^*_{\text{MIP}}\right)} \qquad \textit{(improvement rate)}$$

  These quantities reflect what is the effect of making decision x^*_{MSP} instead of x^*_{MIP}. Large values of Δ_{avg} indicate high penalties for making decision x^*_{MSP} instead of x^*_{MIP} in average scenarios-criteria. Similarly, the larger Δ_{tail} is, the higher benefit is obtained from making decision x^*_{MSP} in tail events. They will be called *deteriorating rate* and *improvement rate*

Models are solved in GAMS 26.1.0 with solver IBM ILOG CPLEX Cplex 12.8.0.0, using a personal computer with an Intel Core i7 processor and 16Gb RAM.

Experiment 1 First experiment will consist on a full factorial design, in which the values of $|I|, |J|, |K|, r, \beta$ fall in these sets:

- $|I| \in \{50, 100, 200\}$
- $|J| \in \{5, 25, 100\}$
- $|K| \in \{3, 6, 9\}$
- $r \in \{0.33, 0.5, 0.67\}$
- $\beta \in \{0.05, 0.1, 0.5\}$

For each tuple (I, J, K) random data will be generated, using algorithm 1, which will then be solved for every pair (r, β). All criteria and scenarios are given same importance and probabilities. That is, $w_k = \frac{1}{|K|}$, $\pi_j = \frac{1}{|J|}$. Time limit was set in two hours by instance, in which all but three of the $3^5 = 243$ configurations were solved to optimality.

Experiment 2 For the next experiment 100 random instances will be created, keeping the values of $|I|, |J|, |K|, r, \beta$ constant and equal to the median value of the previous experiment. That is, $|I| = 100, |J| = 25, |K| = 6, r = 0.5, \beta = 0.1$. All criteria and scenarios are given same importance and probabilities. All 100 instances were solved to optimality.

4.2 Results

Experiment 1 Table 16 shows for each of the 243 instances the solution times of the MSP and the MIP models, and the deteriorating and improvement rates of using the MSP solution instead of the MIP solutions (measured in deviation to MIP solution).

Table 9 shows the correlations between times and rates with the parameters of the instance. It can be seen how the MSP solution has a higher impact when fewer scenarios are considered. In addition to that, it can be appreciated that the MSP solution times decrease when β increase, that is, when more scenarios are included in the β-average computation.

Table 9: Correlations

	|I|	|J|	|K|	r	β
t_{MSP}	0.34	0.09	-0.11	-0.05	-0.19
t_{MIP}	0.51	0.18	-0.14	-0.03	-0.07
Δ_{time}	0.31	0.11	-0.08	-0.02	-0.18
Δ_{avg}	-0.05	-0.57	-0.28	-0.09	-0.36
Δ_{tail}	-0.07	-0.56	-0.18	-0.21	-0.50

This appreciation is confirmed by Table 10, in which it can be seen that the median *time penalty factor* (how many more times does it take to solve the MSP model than the MIP one) is much smaller when $\beta = 0.5$ than when $\beta = 0.05$.

Table 10: Increase on computing times and MSP solution times, grouped by β

	Δ_{time}					t_{MSP}				
β	min	mean	median	max	std	min	mean	median	max	std
0.05	0.94	3188.96	32.77	50473.04	9472.55	0.12	659.49	6.32	7222.95	1787.07
0.10	0.98	1002.35	11.09	20192.48	3245.85	0.12	212.47	2.23	4765.42	728.49
0.50	1.06	19.14	3.75	414.29	55.51	0.13	3.49	0.67	62.14	9.05

Solution times of the MSP model are alarmingly high for some instances, due to the fact that the admissible integrality gap has been set to zero. If that is relaxed, it can be seen that all of the 243 instances reach an integrality gap smaller than 5% in under 3 seconds, 2% in under 5 seconds and 1% in under 88 seconds.

Table 11 groups instances by r and β, and shows the deteriorating and improvement rates. It can be seen that the improvement rate (in the tail) is generally higher than the deteriorating rate (in the average), especially in cases with small r and β.

This claim is also supported with Figure 2, where each of the 243 instances is shown according to the values of Δ_{avg} and Δ_{tail}, and grouped by the values of (r, β). Almost all

Table 11: Values of Δ_{avg} and Δ_{tail}, grouped by r and β

		Δ_{avg}					Δ_{tail}				
r	β	min	mean	median	max	std	min	mean	median	max	std
0.33	0.05	0.03	1.94	1.87	5.68	1.42	0.28	4.37	4.21	9.18	2.43
	0.10	0.02	1.70	1.61	5.68	1.44	0.18	3.54	2.85	9.18	2.42
	0.50	0.00	0.93	0.52	4.46	1.08	0.00	1.57	0.92	4.99	1.46
0.50	0.05	0.03	1.87	1.90	4.30	1.30	0.29	3.58	3.30	6.73	1.89
	0.10	0.02	1.65	1.14	4.30	1.37	0.13	2.87	2.47	6.59	1.86
	0.50	0.00	0.72	0.54	3.51	0.75	0.00	1.07	0.79	3.79	1.01
0.67	0.05	0.03	1.64	1.17	3.93	1.24	0.32	3.04	3.06	6.15	1.62
	0.10	0.01	1.50	1.10	3.93	1.31	0.12	2.43	2.02	5.84	1.58
	0.50	0.00	0.60	0.50	3.16	0.66	0.00	0.80	0.59	3.64	0.81

of the instances ara above the imaginary line $\Delta_{\text{avg}} = \Delta_{\text{tail}}$, which shows that considering the MSP solution improves in the tail more than it loses in the average situations. In addition to that, it can be seen that the largest improvements in the tail are on instances with $\beta = 0.05$ (one of the usual values taken for CVaR), and especially with the smallest values of r. When r and β grow the differences between the MIP and MSP solutions are reduced.

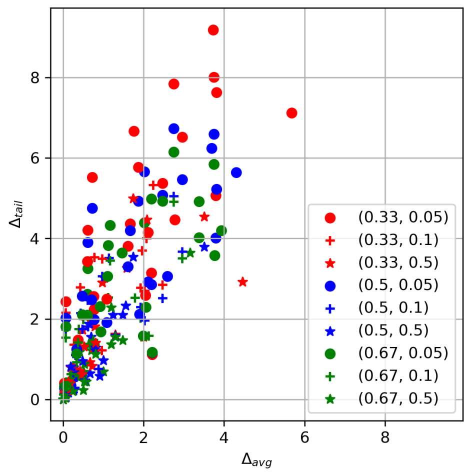


Figure 2: Values Δ_{avg} and Δ_{tail} for each of the 243 instances, grouped by values of (r, β)

Experiment 2 Table 17 contains the results for each of the 100 instances, all of them with constant parameters $|I| = 100, |J| = 25, |K| = 6, r = 0.5, \beta = 0.1$.

Table 12 contains a summary of the results, where it is again seen that the improvements in the tail are better than the loses in the average situations. Although single instances might take a long computing time, the median MSP solution time (3.74s) is definitely satisfactory. It is worth mentioning that the models were implemented without providing any extra bounds or known cuts that could reduce solution times.

Table 12: Summary of set 6

	t_{MSP}	t_{MIP}	Δ_{time}	Δ_{avg}	Δ_{tail}
mean	16.98	0.20	91.31	2.03	3.09
std	46.57	0.03	254.68	1.12	1.49
min	0.53	0.14	2.81	0.16	0.86
25%	1.37	0.17	6.73	1.18	2.09
50%	3.74	0.19	19.72	1.93	2.81
75%	15.50	0.21	86.19	2.52	3.51
max	404.70	0.34	2175.82	5.67	8.57

Finally, figure 3 shows the values of $f_k^j(x)$, where $x = x^*_{\text{MIP}}$ in blue squares and $x = x^*_{\text{MSP}}$ in orange circles, for just one of the created instances. It can be appreciated how x^*_{MIP} performs better than x^*_{MSP} in average criteria-scenarios, but x^*_{MSP} is better with unfavourable situations.

5 Conclusions

In this paper a new concept of solution has been proposed for Multiobjective Stochastic Programming problems, focused on risk-aversion. As such, this concept can be particularly useful in real-life situations where there exists a great concern with respect to unfavourable situations, such as emergency management.

The solution concept is supported by an efficient way to compute it by a Mathematical Programming problem. This model is linear provided that the underlying problem can be linearly representable. Numerical experiments have been conducted for validating this approach, solving a multiobjective stochastic knapsack problem.

The research has also shown that the improvements in the tail (unfavourable situations) are consistently higher than loses on average situations, especially when small values of the parameters β and r are chosen. These differences, although clearly noticeable, are not as high as one could expect. This is possibly due to the randomness of the data. It is reasonable to assume that in actual real-life problems there are choices that are more conservative for every scenario and criterion, and thus being preferable for risk-aversion attitudes.

Results showed that there is a clear increase in computational time; however this is arguably acceptable as a price to pay for being risk-averse. Furthermore, this could also be due to the random nature of the data. Nevertheless, it was also shown that allowing for even rather small integrality gaps (1%) leads to drastic improvement on the computing times.

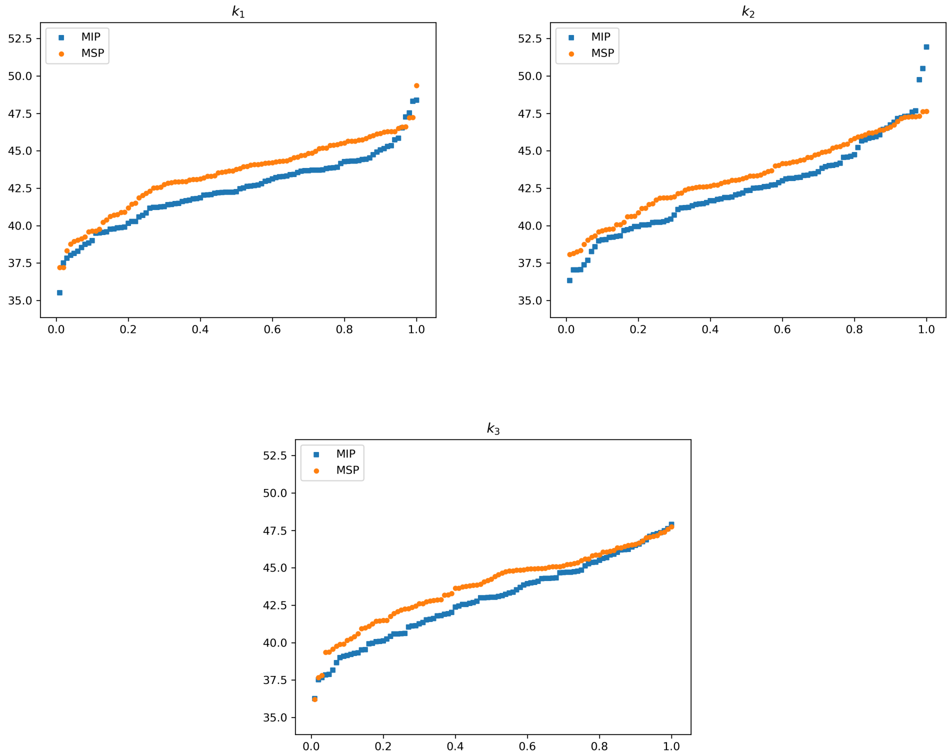


Figure 3: Single instance with 100 scenarios and 3 criteria. For each k, sorted values of $f_k^j(x)$, where $x = x^*_{\text{MIP}}$ in blue squares and $x = x^*_{\text{MSP}}$ in orange circles

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A Extra figures and tables

Table 13: Values of alternative 2 by scenario (j) and criteria (k)

			criteria					
			$w_1 = 0.20$	$w_2 = 0.10$	$w_3 = 0.20$	$w_4 = 0.25$	$w_5 = 0.15$	$w_6 = 0.10$
			k_1	k_2	k_3	k_4	k_5	k_6
scenarios	$\pi_1 = 0.15$	j_1	0.40	0.58	0.39	0.45	0.54	0.18
	$\pi_2 = 0.20$	j_2	0.68	0.74	0.70	0.15	0.54	0.72
	$\pi_3 = 0.30$	j_3	0.93	0.52	0.23	0.82	0.21	0.03
	$\pi_4 = 0.25$	j_4	0.37	0.85	0.07	0.42	0.52	0.22
	$\pi_5 = 0.10$	j_5	0.92	0.13	0.71	0.39	0.90	0.87
β-average, $\beta = 0.30$			0.930	0.832	0.703	0.820	0.660	0.770
r-OWA, $r = 0.17$			0.930					

Table 14: Values of alternative 3 by scenario (j) and criteria (k)

			criteria					
			$w_1 = 0.20$	$w_2 = 0.10$	$w_3 = 0.20$	$w_4 = 0.25$	$w_5 = 0.15$	$w_6 = 0.10$
			k_1	k_2	k_3	k_4	k_5	k_6
scenarios	$\pi_1 = 0.15$	j_1	0.80	0.90	0.61	0.28	0.94	0.09
	$\pi_2 = 0.20$	j_2	0.29	0.48	0.26	0.23	0.21	0.07
	$\pi_3 = 0.30$	j_3	0.73	0.65	0.32	0.56	0.95	0.65
	$\pi_4 = 0.25$	j_4	0.58	0.39	0.21	0.66	0.70	0.93
	$\pi_5 = 0.10$	j_5	0.73	0.22	0.33	0.31	0.32	0.38
β-average, $\beta = 0.30$			0.765	0.775	0.468	0.643	0.950	0.883
r-OWA, $r = 0.17$			0.943					

Table 15: Values of alternative 4 by scenario (j) and criteria (k)

			criteria					
			$w_1 = 0.20$	$w_2 = 0.10$	$w_3 = 0.20$	$w_4 = 0.25$	$w_5 = 0.15$	$w_6 = 0.10$
			k_1	k_2	k_3	k_4	k_5	k_6
scenarios	$\pi_1 = 0.15$	j_1	0.30	0.52	0.12	0.68	0.46	0.73
	$\pi_2 = 0.20$	j_2	1.00	0.57	0.46	0.82	0.90	0.72
	$\pi_3 = 0.30$	j_3	0.18	0.76	0.30	0.34	0.54	0.99
	$\pi_4 = 0.25$	j_4	0.53	0.21	0.13	0.12	0.66	0.86
	$\pi_5 = 0.10$	j_5	0.98	0.46	0.50	0.29	0.27	0.40
β-average, $\beta = 0.30$			0.993	0.760	0.473	0.773	0.820	0.990
r-OWA, $r = 0.17$			0.993					

Table 16: All instances of first experiment. The three instances with 200 objects, 100 scenarios, 6 criteria and $\beta = 0.05$ did not reach the optimal solution in 2 hours. The integrality gaps of the solution shown are $0.31\%, 0.24\%$ and 0.19% for $r = 0.33, 0.5$ and 0.67 respectively

		$\beta \rightarrow$	0.05												0.1												0.5											
		$r \rightarrow$	0.33				0.5				0.67				0.33				0.5				0.67				0.33				0.5				0.67			
|I|	|J|	|K|	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}
50	5	3	0.12	0.13	3.75	8.01	0.15	0.12	3.75	8.01	0.18	0.14	3.51	4.54	0.14	0.14	3.75	6.59	0.12	0.12	3.75	6.59	0.20	0.17	3.51	3.79	0.12	0.12	3.75	5.84	0.12	0.12	3.75	5.84	0.13	0.12	3.16	3.64
		6	0.22	0.11	3.79	5.07	0.25	0.11	3.79	5.07	0.18	0.11	1.03	3.01	0.30	0.13	3.79	4.01	0.25	0.13	3.79	4.01	0.23	0.15	1.48	2.10	0.23	0.16	3.77	3.58	0.23	0.14	3.77	3.58	0.18	0.17	1.48	1.47
		9	0.28	0.14	1.76	6.67	0.36	0.14	1.76	6.67	0.20	0.15	1.58	3.26	0.22	0.14	2.02	5.66	0.21	0.14	2.02	5.66	0.22	0.14	1.24	2.10	0.20	0.15	2.02	4.39	0.22	0.13	2.02	4.39	0.25	0.15	1.20	1.37
	25	3	0.76	0.18	2.47	5.37	0.66	0.17	2.47	2.85	0.26	0.17	2.00	1.55	0.64	0.18	2.47	5.08	0.62	0.16	2.47	2.51	0.25	0.15	1.00	0.97	0.60	0.17	2.47	4.93	0.51	0.16	1.79	2.52	0.26	0.14	1.00	0.69
		6	1.20	0.16	1.87	5.77	0.91	0.29	1.96	3.70	0.49	0.18	0.79	1.81	1.15	0.18	1.87	4.93	0.74	0.18	1.14	3.51	0.41	0.18	0.70	1.55	0.93	0.16	1.17	4.33	0.78	0.18	1.17	3.45	0.38	0.16	0.71	1.22
		9	0.69	0.16	0.61	4.21	0.78	0.16	0.43	2.78	0.57	0.16	0.67	0.92	0.52	0.15	0.61	3.90	0.96	0.16	0.44	2.14	0.61	0.16	0.67	0.65	0.69	0.15	0.61	3.25	1.02	0.18	0.39	1.75	0.47	0.18	0.51	0.68
	100	3	1.15	0.15	0.07	2.43	0.78	0.15	0.07	2.15	0.44	0.14	0.07	0.16	1.07	0.14	0.07	2.02	0.83	0.19	0.07	1.74	0.34	0.14	0.07	0.16	1.14	0.15	0.07	1.81	0.85	0.16	0.07	1.53	0.36	0.14	0.07	0.14
		6	2.51	0.20	0.77	2.24	4.45	0.22	0.78	1.19	3.52	0.20	0.23	0.47	2.62	0.25	0.77	1.99	5.09	0.18	0.31	1.10	5.45	0.20	0.19	0.29	2.70	0.22	0.77	1.38	3.31	0.19	0.47	0.63	5.59	0.19	0.23	0.27
		9	4.06	0.18	0.03	0.41	1.47	0.17	0.03	0.30	1.07	0.16	0.00	0.00	2.75	0.17	0.03	0.29	1.12	0.16	0.03	0.30	1.11	0.16	0.00	0.00	2.06	0.19	0.03	0.32	1.10	0.16	0.03	0.18	1.16	0.17	0.00	0.00
100	5	3	1.24	0.26	3.81	7.63	1.29	0.20	3.81	7.63	0.37	0.22	2.08	4.47	0.72	0.18	3.81	5.22	0.66	0.18	3.81	5.22	0.32	0.27	1.56	2.33	0.74	0.19	3.38	4.02	1.13	0.23	3.38	4.02	0.28	0.20	0.63	1.36
		6	8.68	0.22	5.68	7.12	8.69	0.19	5.68	7.12	0.43	0.17	4.46	2.92	0.65	0.20	4.30	5.64	1.06	0.19	4.30	5.64	0.35	0.17	0.81	1.28	1.18	0.22	3.93	4.20	0.64	0.18	3.93	4.20	0.28	0.18	0.53	0.88
		9	3.31	0.18	2.19	3.14	3.26	0.20	2.19	3.14	0.67	0.18	0.97	2.90	1.04	0.20	2.19	2.86	0.96	0.20	2.19	2.86	0.23	0.14	0.64	1.88	1.02	0.17	0.92	2.31	0.90	0.17	0.92	2.31	0.24	0.18	0.41	1.18
	25	3	10.65	0.17	2.96	6.52	3.39	0.18	2.07	4.00	0.29	0.15	0.48	1.73	7.09	0.18	2.96	5.46	1.83	0.19	2.96	3.67	0.30	0.14	0.48	0.95	3.46	0.19	2.19	4.98	1.30	0.16	2.96	3.50	0.34	0.16	0.41	0.59
		6	32.12	0.20	2.78	4.47	9.18	0.19	0.78	3.53	0.44	0.18	0.52	1.31	26.53	0.18	2.59	3.06	3.64	0.22	0.61	2.47	0.32	0.15	0.26	0.79	12.77	0.17	0.60	2.62	0.90	0.17	0.50	2.00	0.41	0.17	0.26	0.65
		9	8.58	0.18	0.72	5.52	1.90	0.17	0.97	3.49	0.42	0.18	0.24	0.82	6.32	0.19	0.72	4.75	1.24	0.16	0.97	3.06	0.51	0.20	0.24	0.44	1.60	0.19	1.12	3.83	0.88	0.19	1.12	2.53	0.59	0.17	0.50	0.23
	100	3	51.23	0.22	2.21	1.12	1.67	0.21	0.27	1.36	0.82	0.18	0.09	0.25	22.70	0.21	2.21	1.16	1.22	0.19	0.34	1.05	0.81	0.18	0.05	0.17	18.75	0.16	2.21	1.17	0.84	0.22	0.34	0.92	0.75	0.18	0.05	0.13
		6	48.25	0.18	0.76	2.56	31.87	0.17	0.62	2.05	62.14	0.15	0.42	0.70	24.73	0.18	0.71	2.48	27.08	0.18	0.62	1.81	42.18	0.19	0.28	0.55	20.26	0.17	0.60	2.10	22.09	0.20	0.75	1.48	7.79	0.20	0.17	0.50
		9	2.16	0.19	0.37	1.48	3.34	0.18	0.29	0.77	1.84	0.17	0.18	0.41	1.80	0.17	0.34	1.25	2.87	0.20	0.28	0.69	2.22	0.19	0.26	0.22	1.67	0.18	0.34	1.14	2.77	0.19	0.20	0.63	3.09	0.16	0.08	0.13
200	5	3	146.24	0.23	1.61	3.81	140.12	0.20	1.61	3.81	7.71	0.23	1.30	1.61	151.22	0.21	1.61	3.30	135.09	0.24	1.61	3.30	4.60	0.21	1.30	1.58	83.44	0.22	1.10	3.06	89.20	0.21	1.10	3.06	4.21	0.22	1.30	1.55
		6	88.70	0.19	1.08	2.50	89.69	0.19	1.08	2.50	5.14	0.17	0.72	0.83	96.44	0.19	1.08	1.91	91.66	0.18	1.08	1.91	2.93	0.18	0.91	0.58	39.26	0.18	0.94	1.68	32.92	0.18	0.94	1.68	0.70	0.17	0.58	0.44
		9	468.37	0.15	3.73	9.18	484.89	0.14	3.73	9.18	29.46	0.16	1.74	4.99	304.04	0.16	3.69	6.24	305.90	0.16	3.69	6.24	2.71	0.16	1.74	3.54	110.03	0.15	3.38	4.92	107.34	0.14	3.38	4.92	0.91	0.17	1.20	2.28
	25	3	5629.58	0.33	2.75	7.84	4765.42	0.24	2.24	5.33	4.86	0.24	0.81	1.40	5430.90	0.25	2.75	6.73	3394.56	0.24	2.75	5.05	5.32	0.28	0.81	1.22	6896.05	0.25	2.75	6.15	2546.43	0.21	2.75	4.91	5.66	0.34	0.81	1.13
		6	2886.13	0.19	1.67	4.36	146.48	0.17	1.93	2.77	0.57	0.17	0.19	0.79	1651.91	0.21	1.67	4.20	15.06	0.22	1.93	2.29	0.71	0.21	0.19	0.81	93.66	0.19	1.46	3.64	19.36	0.19	1.93	2.02	0.55	0.18	0.12	0.40
		9	1235.12	0.32	2.05	2.59	342.32	0.22	0.96	1.22	1.99	0.21	0.22	0.26	404.70	0.29	1.90	2.12	28.09	0.21	0.88	0.76	0.82	0.20	0.13	0.17	99.73	0.21	1.99	1.58	2.23	0.22	0.39	0.58	0.87	0.20	0.06	0.08
	100	3	703.05	0.23	2.11	4.15	373.65	0.22	2.03	2.70	1.42	0.22	0.47	0.63	731.09	0.22	2.11	2.92	157.29	0.20	2.03	1.96	1.11	0.22	0.54	0.47	596.88	0.27	2.06	2.29	349.78	0.30	2.13	1.58	3.22	0.29	0.53	0.44
		6	7222.95	0.22	0.60	3.43	1814.25	0.18	0.48	2.08	22.11	0.21	0.13	0.44	7217.64	0.14	0.47	2.57	916.42	0.21	0.48	1.75	7.04	0.24	0.28	0.27	7216.94	0.15	0.47	2.11	656.48	0.20	0.37	1.41	7.89	0.22	0.24	0.20
		9	3321.23	0.34	0.07	0.28	16.40	0.20	0.02	0.18	2.33	0.17	0.08	0.08	198.14	0.19	0.07	0.32	14.84	0.21	0.02	0.13	2.31	0.21	0.08	0.05	47.16	0.23	0.08	0.33	9.77	0.21	0.01	0.12	2.63	0.20	0.06	0.04

Table 17: All instances of second experiment. $|I| = 100, |J| = 25, |K| = 6, r = 0.5, \beta = 0.1$

t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}	t_{MSP}	t_{MIP}	Δ_{avg}	Δ_{tail}
31.15	0.23	1.53	3.20	2.15	0.16	2.24	2.09
1.92	0.21	1.66	6.17	20.09	0.16	1.80	6.14
8.75	0.24	0.52	3.07	7.18	0.16	2.13	1.93
28.06	0.23	5.08	2.86	1.02	0.16	3.03	3.61
1.36	0.30	1.00	1.80	3.58	0.24	1.81	6.12
3.67	0.20	2.27	2.50	3.64	0.19	1.19	3.07
2.00	0.20	2.51	2.03	128.69	0.23	3.27	2.98
192.11	0.16	2.61	8.23	0.89	0.18	1.45	0.93
0.94	0.20	0.43	2.23	1.62	0.23	1.85	3.56
0.80	0.18	1.64	2.55	4.19	0.22	2.10	1.97
16.40	0.19	2.23	2.45	2.16	0.19	0.16	1.46
1.21	0.18	2.82	1.50	1.46	0.24	2.48	2.00
1.79	0.20	0.72	2.77	0.69	0.20	1.79	2.54
21.78	0.21	4.50	4.61	20.73	0.20	2.26	3.50
1.35	0.19	0.69	0.86	1.86	0.24	1.77	2.63
31.11	0.19	0.98	3.21	14.92	0.17	1.99	8.57
8.44	0.19	1.82	3.81	0.78	0.20	0.85	1.92
1.75	0.21	0.88	0.92	10.48	0.23	2.50	2.29
1.94	0.21	2.18	2.65	1.63	0.24	2.08	2.29
0.98	0.20	0.87	3.27	10.78	0.18	0.34	1.80
27.72	0.22	2.03	5.20	38.80	0.20	1.96	4.69
14.72	0.15	3.34	0.99	19.74	0.24	0.65	2.20
0.67	0.24	0.81	2.69	1.37	0.30	2.82	2.92
3.54	0.20	2.64	2.75	6.28	0.19	2.02	2.08
6.37	0.21	2.79	6.35	22.27	0.34	1.91	3.13
1.86	0.23	0.93	2.09	1.69	0.20	2.21	2.42
1.54	0.20	2.00	3.45	27.77	0.19	0.76	3.28
40.16	0.17	2.06	3.44	2.00	0.21	2.57	1.93
7.23	0.21	3.17	3.17	2.61	0.18	2.14	3.33
5.77	0.17	2.84	1.98	40.93	0.18	1.53	4.61
2.10	0.19	1.39	3.00	18.84	0.16	0.89	4.37
404.70	0.19	1.50	2.82	11.26	0.16	3.98	4.76
24.26	0.18	4.81	3.42	14.41	0.18	1.82	5.87
0.76	0.20	1.28	3.88	12.14	0.16	2.75	2.75
0.64	0.20	0.87	1.39	12.58	0.17	1.42	3.46
0.97	0.23	1.77	2.19	0.84	0.18	0.41	2.15
0.53	0.18	1.95	2.04	5.20	0.19	3.80	2.02
7.24	0.22	2.21	1.68	28.16	0.15	4.68	3.56
0.87	0.25	0.71	1.42	39.10	0.16	3.47	3.59
8.51	0.20	2.48	4.06	19.22	0.16	3.78	3.13
13.06	0.20	4.44	2.80	0.56	0.20	0.63	3.22
59.78	0.20	5.67	4.91	0.68	0.17	1.92	2.44
67.50	0.19	2.96	3.02	0.70	0.17	1.86	1.40
3.80	0.17	0.79	1.20	15.20	0.17	0.78	2.66
3.25	0.20	2.24	1.57	0.88	0.17	2.31	2.13
5.23	0.16	1.14	4.91	0.59	0.15	1.78	1.68
0.71	0.17	0.89	3.01	1.08	0.20	2.21	3.14
4.19	0.17	3.09	2.32	1.14	0.17	0.76	2.48
3.53	0.18	1.37	6.33	1.58	0.19	1.45	3.14
19.99	0.14	3.48	5.05	13.48	0.18	1.71	5.41